

PHASELESS SAMPLING AND RECONSTRUCTION OF REAL-VALUED SIGNALS IN SHIFT-INVARIANT SPACES

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ABSTRACT. Sampling in shift-invariant spaces is a realistic model for signals with smooth spectrum. In this paper, we consider phaseless sampling and reconstruction of real-valued signals in a shift-invariant space from their magnitude measurements on the whole Euclidean space and from their phaseless samples taken on a discrete set with finite sampling density. We introduce an undirected graph to a signal and use connectivity of the graph to characterize whether the signal can be determined, up to a sign, from its magnitude measurements on the whole Euclidean space. Under the local complement property assumption on a shift-invariant space, we find a discrete set with finite sampling density such that signals in the shift-invariant space, that are determined from their magnitude measurements on the whole Euclidean space, can be reconstructed in a stable way from their phaseless samples taken on that discrete set. In this paper, we also propose a reconstruction algorithm which provides a suboptimal approximation to the original signal when its noisy phaseless samples are available only. Finally, numerical simulations are performed to demonstrate the robust reconstruction of box spline signals from their noisy phaseless samples.

1. INTRODUCTION

In this paper, we consider the phaseless sampling and reconstruction problem whether a real-valued signal f on \mathbb{R}^d can be determined, up to a sign, from its magnitude measurements $|f(x)|$ on \mathbb{R}^d or a subset $X \subset \mathbb{R}^d$. The above problem is ill-posed inherently and it could be solved only if we have some extra information about the signal f .

The additional knowledge about the signals in this paper is that they live in a shift-invariant space

$$(1.1) \quad V(\phi) := \left\{ \sum_{k \in \mathbb{Z}^d} c(k) \phi(x - k) : c(k) \in \mathbb{R} \text{ for all } k \in \mathbb{Z}^d \right\}$$

generated by a real-valued continuous function ϕ with compact support. Shift-invariant spaces have been used in wavelet analysis and approximation theory, and sampling in shift-invariant spaces is a realistic model for signals with smooth spectrum, see [4, 6, 11, 17, 30] and references therein.

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Typical examples of shift-invariant spaces include those generated by refinable functions ([16, 33]) and box splines M_{Ξ} , which are defined by

$$(1.2) \quad \int_{\mathbb{R}^d} g(x) M_{\Xi}(x) dx = \int_{\mathbb{R}^s} g(\Xi y) dy, \quad g \in L^2(\mathbb{R}^d),$$

where $\Xi \in \mathbb{Z}^{d \times s}$ is a matrix with full rank d ([19, 45, 46]).

The phaseless sampling and reconstruction problem of one-dimensional signals in shift-invariant spaces has been studied in [20, 36, 37, 40, 44]. Thakur proved in [44] that one-dimensional real-valued signals in a Paley-Wiener space, the shift-invariant space generated by the sinc function $\frac{\sin \pi t}{\pi t}$, could be reconstructed from their phaseless samples taken at more than twice the Nyquist rate. Reconstruction of one-dimensional signals in a shift-invariant space was studied in [40] when frequency magnitude measurements are available. Not all signals in a shift-invariant space generated by a compactly supported function are determined, up to a sign, from their magnitude measurements on the whole line. In [20], the set of signals that can be determined from their magnitude measurement on the real line \mathbb{R} is fully characterized and a fast algorithm is proposed to reconstruct signals in a shift-invariant space from their phaseless samples taken on a discrete set with finite sampling density. Up to our knowledge, there is no literature available on the phaseless sampling and reconstruction of high-dimensional signals in a shift-invariant space, which is the core of this paper.

The phaseless sampling and reconstruction of signals in a shift-invariant space is an infinite-dimensional phase retrieval problem, which has received considerable attention in recent years [1, 2, 3, 12, 20, 34, 36, 37, 40, 44]. Phase retrieval plays important roles in signal/speech/image processing and it is a highly nonlinear mathematical problem. Even in the finite-dimensional setting, there are still lots of mathematical and engineering problems about phase retrieval unanswered. The reader may refer to [7, 8, 13, 14, 15, 25, 28, 32, 39] and references therein for historical remarks and recent advances.

The paper is organized as follows. An introductory problem about phaseless sampling and reconstruction in the shift-invariant space $V(\phi)$ is that a real-valued signal f is determined, up to a sign, from its magnitude $|f(x)|, x \in \mathbb{R}^d$. An equivalence has been provided in [20], see Theorem 2.1 in Section 2, that the signal f must be nonseparable, i.e., f is not the sum of two nonzero signals in $V(\phi)$ with their supports being essentially disjoint. A natural question arisen is how to determine nonseparability of a given signal in a shift-invariant space. For a one-dimensional signal f , it is shown in [20] that f is nonseparable if and only if the amplitude vector $(c(k))_{k \in \mathbb{Z}}$ does not have consecutive zeros. However, there is no corresponding notion of consecutive zeros in the high-dimensional shift-invariant spaces $V(\phi)$ with $d \geq 2$. In Section 2, we introduce an undirected graph \mathcal{G}_f to a high-dimensional signal f in the shift-invariant space $V(\phi)$ and use the connectivity of the graph \mathcal{G}_f to characterize the nonseparability of the signal

f , i.e., it is determined, up to a sign, from the magnitude measurements $|f(x)|, x \in \mathbb{R}^d$, see Theorems 2.3 and 2.5.

In Section 3, we consider the preparatory problem whether a signal in a shift-invariant space is determined, up to a sign, from its phaseless samples taken on a discrete set with finite sampling density. A necessary condition is that the signal is nonseparable. In Theorem 3.1, we show that the above nonseparable requirement is also sufficient, and furthermore the discrete sampling set can be selected explicitly to be of the form $\Gamma + \mathbb{Z}^d$, where Γ contains finitely many elements. However, the above result does not provide us an algorithm to reconstruct a nonseparable signal from its phaseless samples taken on that discrete set. In Section 3, we introduce a local complement property for the shift-invariant space $V(\phi)$, which is similar to the complement property for frames in Hilbert/Banach spaces [3, 7, 9, 12, 20]. Local complement property is closely related to local phase retrievability, see Appendix A. We apply the local complement property in Theorem 3.4 to find another discrete set with finite sampling density on which nonseparable signals in a shift-invariant space can be recovered from their phaseless samples. Moreover, our proof of Theorem 3.4 is constructive.

Stability is a pivotal problem in the phaseless sampling and reconstruction. We aim to find a good approximation to the original signal f , up to a sign, when its noisy phaseless samples

$$z_\epsilon(y) = |f(y)| + \epsilon(y), \quad y \in X,$$

taken on a discrete set X with finite sampling density are available only, where $\epsilon = (\epsilon(y))_{y \in X}$ is the additive bounded noise. A conventional approach of the above problem is to solve the following minimization problem,

$$\min \sum_{y \in X} \left| |g(y)| - z_\epsilon(y) \right|^2 \text{ subject to } g \in V(\phi).$$

However, it is infinite-dimensional and infeasible. For a shift-invariant sampling set X of the form $\bigcup_{m=1}^M \Gamma_m + \mathbb{Z}^d$, we propose a four-step approach starting from local minimization problems,

$$\min \sum_{y \in \Gamma_m + l} \left| |g(y)| - z_\epsilon(y) \right|^2 \text{ subject to } g \in V(\phi),$$

where $1 \leq m \leq M$ and $l \in \mathbb{Z}^d$, see (4.5)–(4.9) in Section 4. In Theorem 4.1, we establish stability of the four-step approach to reconstruct nonseparable signals in a shift-invariant space. The above stability implies the nonexistence of resonance phenomenon when the noise level is far below the minimal magnitude of amplitude vector of the original signal, see Remark 4.2.

A fundamental problem in the phaseless sampling and reconstruction is to design efficient and robust algorithms for signal reconstruction in a noisy environment. Based on the four-step approach in Section 4, we propose an algorithm to reconstruct nonseparable signals in $V(\phi)$ from their noisy phaseless samples on the shift-invariant set $\bigcup_{m=1}^M \Gamma_m + \mathbb{Z}^d$. The complexity

of the proposed algorithm depends almost linearly on the support length of the original nonseparable signal. The reader may refer to [13, 14, 24, 26, 27, 39] and references therein for various algorithms to reconstruct a finite-dimensional signals from magnitude of its finite frame measurements. The implementation and performance of the proposed algorithm to recover box spline signals are given in Section 5.

Proofs are collected in Section 6 and the local complement property is discussed in Appendix A.

Notation: Denote the cardinality of a set E by $\#E$ and the closed ball in \mathbb{R}^d with center x and radius $R \geq 0$ by $B(x, R)$. Define the power $x^k = \prod_{i=1}^d x_i^{k_i}$ for $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$ and $k = (k_1, \dots, k_d)^T \in \mathbb{Z}^d$, and the partial order $x \leq y$ for $y = (y_1, \dots, y_d)^T \in \mathbb{R}^d$ if $x_i \leq y_i, 1 \leq i \leq d$.

2. PHASE RETRIEVABILITY, NONSEPARABILITY AND CONNECTIVITY

The phase retrievability of a real-valued signal on \mathbb{R}^d is whether it is determined, up to a sign, from its magnitude measurements. It is characterized in [20] as follows.

Theorem 2.1. *Let ϕ be a real-valued continuous function with compact support, and $V(\phi)$ be the shift-invariant space in (1.1) generated by ϕ . Then a signal $f \in V(\phi)$ is determined, up to a sign, by its magnitude measurements $|f(x)|, x \in \mathbb{R}^d$, if and only if f is nonseparable, i.e., there does not exist nonzero signals f_1 and f_2 in $V(\phi)$ such that*

$$(2.1) \quad f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0.$$

The question arisen is how to determine nonseparability of a signal in a shift-invariant space. To answer the above question, we need the one-to-one correspondence between an amplitude vector c and a signal f in the shift-invariant space $V(\phi)$,

$$(2.2) \quad c := (c(k))_{k \in \mathbb{Z}^d} \mapsto \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k) =: f \in V(\phi),$$

which is known as the *global linear independence* of the generator ϕ [10, 30, 38]. For $d = 1$, the nonseparability of a signal in a shift-invariant space is characterized in [20] that its amplitude vector does not have consecutive zeros. However, there is no corresponding notion of consecutive zeros in the high-dimensional setting ($d \geq 2$). To characterize the nonseparability of signals on \mathbb{R}^d with $d \geq 2$, we introduce an undirected graph for a signal in the shift-invariant space $V(\phi)$ generated by a real-valued continuous function ϕ with compact support.

Definition 2.2. For any $f(x) = \sum_{k \in \mathbb{Z}^d} c(k) \phi(x - k) \in V(\phi)$, define an *undirected graph*

$$(2.3) \quad \mathcal{G}_f := (V_f, E_f),$$

where the vertex set

$$V_f = \{k \in \mathbb{Z}^d : c(k) \neq 0\}$$

contains supports of the amplitude vector of the signal f , and

$$E_f = \{(k, k') \in V_f \times V_f : k \neq k' \text{ and } \phi(x-k)\phi(x-k') \neq 0 \text{ for some } x \in \mathbb{R}^d\}$$

is the edge set associated with the signal f .

The graph \mathcal{G}_f in (2.3) is well-defined for any signal f in the shift-invariant space $V(\phi)$ when ϕ has global linear independence. Moreover,

$$(2.4) \quad (k, k') \in E_f \text{ if and only if } k - k' \in \Lambda_\phi,$$

where Λ_ϕ contains all $k \in \mathbb{Z}^d$ such that

$$(2.5) \quad S_k := \{x : \phi(x)\phi(x-k) \neq 0\} \neq \emptyset.$$

In the following theorem, we show that connectivity of the graph \mathcal{G}_f is a necessary condition for the nonseparability of the signal $f \in V(\phi)$.

Theorem 2.3. *Let ϕ be a compactly supported continuous function on \mathbb{R}^d with global linear independence, and $V(\phi)$ be the shift-invariant space (1.1) generated by ϕ . If $f \in V(\phi)$ is nonseparable, then the graph \mathcal{G}_f in (2.3) is connected.*

Before stating sufficiency for the connectivity of the graph \mathcal{G}_f , we recall a concept of local linear independence on an open set.

Definition 2.4. Let ϕ be a continuous function with compact support and A be an open set. We say that ϕ has *local linear independence on A* if $\sum_{k \in \mathbb{Z}^d} c(k)\phi(x-k) = 0$ for all $x \in A$ implies that $c(k) = 0$ for all $k \in \mathbb{Z}^d$ satisfying $\phi(x-k) \not\equiv 0$ on A .

The global linear independence of a compactly supported function ϕ can be interpreted as its local linear independence on \mathbb{R}^d ([10, 43]). Define

$$(2.6) \quad \Phi_A(x) := (\phi(x-k))_{k \in K_A}, \quad x \in A$$

and

$$(2.7) \quad K_A := \{k \in \mathbb{Z}^d : \phi(\cdot - k) \not\equiv 0 \text{ on } A\}.$$

One may verify that ϕ has local linear independence on A if and only if the dimension of the linear space spanned by $\Phi_A(x), x \in A$, is the cardinality of the set K_A . The above characterization can be used to verify the local linear independence on a bounded open set, especially when ϕ has the explicit expression. For instance, one may verify that the generator ϕ_0 in Example 2.7 below has local linear independence on $(0, 1)$, but it is locally linearly dependent on $(0, 1/2)$ and $(1/2, 1)$.

The local linear independence on any open sets and global linear independence are equivalent for some compactly supported functions, such as box splines and one-dimensional refinable functions ([18, 22, 23, 29, 41]). In the following theorem, we show that the converse in Theorem 2.3 is also true if

the generator ϕ is assumed to have local linear independence on any open set.

Theorem 2.5. *Let ϕ be a compactly supported continuous function on \mathbb{R}^d with local linear independence on any open set, and f be a signal in the shift-invariant space $V(\phi)$. If the graph \mathcal{G}_f in (2.3) is connected, then f is nonseparable.*

For $d = 1$, we have

$$(2.8) \quad (k, k') \in E_f \text{ if and only if } |k - k'| \leq L - 1,$$

provided that the support of ϕ is $[0, L]$ for some $L \geq 1$. This together with Theorems 2.3 and 2.5 leads to the following result, which is established in [20] under different assumptions on the generator ϕ .

Corollary 2.6. *Let ϕ be a compactly supported continuous function on \mathbb{R} , and $f = \sum_{k \in \mathbb{Z}} c(k)\phi(\cdot - k)$ be a signal in the shift-invariant space $V(\phi)$. If ϕ has local linear independence on any open set and its supporting set is $[0, L]$ for some $L \geq 1$, then f is nonseparable if and only if $\sum_{l=0}^{L-2} |c(k+l)|^2 \neq 0$ for all $K_-(f) - L + 1 < k < K_+(f) + 1$, where $K_-(f) = \inf\{k : c(k) \neq 0\}$ and $K_+(f) = \sup\{k : c(k) \neq 0\}$.*

As demonstrated by the following example, the connectivity of the graph \mathcal{G}_f is not sufficient for the signal f to be nonseparable if the local linear independence assumption on the generator ϕ is dropped.

Example 2.7. Define $\phi_0(t) = h(4t-1) + h(4t-3) + h(4t-5) - h(4t-7)$, where $h(t) = \max(1-|t|, 0)$ is the hat function supported on $[-1, 1]$. One may easily verify that ϕ_0 is a continuous function having global linear independence. Set

$$f_1(t) = \sum_{k \in \mathbb{Z}} \phi_0(t - k) \quad \text{and} \quad f_2(t) = \sum_{k \in \mathbb{Z}} (-1)^k \phi_0(t - k).$$

Then f_1 and f_2 are nonzero signals in $V(\phi_0)$ supported on $[0, 1/2] + \mathbb{Z}$ and $[1/2, 1] + \mathbb{Z}$ respectively, and $f_1(t)f_2(t) = 0$ for all $t \in \mathbb{R}$. Hence $f_1 \pm 2f_2$ have the same magnitude measurement $|f_1| + 2|f_2|$ on the real line but they are different, even up to a sign, i.e., $f_1 + 2f_2 \not\equiv \pm(f_1 - 2f_2)$. On the other hand, one may verify that their associated graphs $\mathcal{G}_{f_1 \pm 2f_2}$ are connected.

Consider a continuous solution ϕ of a refinement equation

$$(2.9) \quad \phi(x) = \sum_{n=0}^N a(n)\phi(2x - n) \quad \text{and} \quad \int_{\mathbb{R}} \phi(x)dx = 1$$

with global linear independence, where $\sum_{n=0}^N a(n) = 2$ and $N \geq 1$ ([16, 33]). The B-spline B_N of order N , which is obtained by convolving the indicator function $\chi_{[0,1]}$ on the unit interval N times, satisfies the above refinement equation ([45, 46]). The function ϕ in (2.9) has support $[0, N]$ and it has local linear independence on any open set if and only if it has global linear independence ([31, 35, 41]). Therefore we have the following

result for wavelet signals by Theorems 2.3 and 2.5, which is also established in [20] with a different approach.

Corollary 2.8. *Let ϕ satisfy the refinement equation (2.9) and have global linear independence. Then $f \in V(\phi)$ is nonseparable if and only if the graph \mathcal{G}_f in (2.3) is connected.*

The local linear independence requirement in Theorem 2.5 can be verified for box splines M_Ξ in (1.2). It is known that the box spline M_Ξ has local linear independence on any open set if and only if all $d \times d$ submatrices of Ξ have determinants being either 0 or ± 1 if and only if it has global linear independence ([18, 22, 23, 29]). The reader may refer to [19] for more properties and applications of box splines. As applications of Theorems 2.3 and 2.5, we have the following result for box spline signals.

Corollary 2.9. *Let $\Xi \in \mathbb{Z}^{d \times s}$ be a matrix of full rank d such that its $d \times d$ submatrices have determinants being either 0 or ± 1 . Then $f \in V(M_\Xi)$ is nonseparable if and only if the graph \mathcal{G}_f in (2.3) is connected.*

3. PHASELESS SAMPLING AND RECONSTRUCTION

In this section, we consider the problem whether a signal in the shift-invariant space $V(\phi)$ is determined, up to a sign, from its phaseless samples taken on a discrete set with finite sampling density. Here we define the sampling density of a discrete set $X \subset \mathbb{R}^d$ by

$$D(X) := D_+(X) = D_-(X)$$

if its upper sampling density $D_+(X)$ and lower sampling density $D_-(X)$ are the same [4, 21, 42], where

$$(3.1) \quad D_+(X) := \limsup_{R \rightarrow +\infty} \sup_{x \in \mathbb{R}^d} \frac{\#(X \cap B(x, R))}{R^d}$$

and

$$(3.2) \quad D_-(X) := \liminf_{R \rightarrow +\infty} \inf_{x \in \mathbb{R}^d} \frac{\#(X \cap B(x, R))}{R^d}.$$

One may easily verify that a shift-invariant set $X = \Gamma + \mathbb{Z}^d$ generated by a finite set Γ has sampling density $\#\Gamma$.

To determine a signal, up to a sign, from its phaseless samples taken on a discrete set, a necessary condition is that the signal is nonseparable (hence phase retrievable). In the next theorem, we show that the above requirement is also sufficient.

Theorem 3.1. *Let ϕ be a compactly supported continuous function and $V(\phi)$ be the shift-invariant space in (1.1) generated by ϕ . Then there exists a discrete set $\Gamma \subset (0, 1)^d$ such that any nonseparable signal $f \in V(\phi)$ is determined, up to a sign, from its phaseless samples on the set $\Gamma + \mathbb{Z}^d$ with finite sampling density.*

For a compactly supported function ϕ and a bounded open set A , let

$$(3.3) \quad W_A \text{ be the linear space spanned by } \Phi_A(x)(\Phi_A(x))^T, x \in A,$$

where Φ_A is given in (2.6). Observe that for any bounded set A , the space W_A spanned by outer products $\Phi_A(x)(\Phi_A(x))^T, x \in A$, is of finite dimension. Therefore there exists a finite set $\Gamma \subset A$ such that outer products $\Phi_A(\gamma)(\Phi_A(\gamma))^T, \gamma \in \Gamma$, is a basis of the linear space W_A . In the proof of Theorem 3.1, we use $A = (0, 1)^d$ and apply the above procedure to select the finite set Γ . With the above selection of the set Γ ,

$$(3.4) \quad \#\Gamma = \dim W_{(0,1)^d},$$

and $|f(x)|^2, x \in \mathbb{R}^d$, are determined by $|f(\gamma)|^2, \gamma \in \Gamma + \mathbb{Z}^d$, see Section 6.2 for the detailed proof.

As symmetric matrices in the space $W_{(0,1)^d}$ are of size $\#K_{(0,1)^d}$, we have the following result about the sampling density.

Corollary 3.2. *Let ϕ and $V(\phi)$ be as in Theorem 3.1. Then any non-separable signal $f \in V(\phi)$ is determined from its phaseless samples on a shift-invariant set $\Gamma + \mathbb{Z}^d$ with sampling density*

$$D(\Gamma + \mathbb{Z}^d) \leq \dim W_{(0,1)^d} \leq \frac{1}{2} \#K_{(0,1)^d} (\#K_{(0,1)^d} + 1),$$

where $K_{(0,1)^d}$ is in (2.7).

The explicit construction of a discrete set with finite sampling density in Theorem 3.1 does not provide an algorithm to reconstruct a nonseparable signal from its phaseless samples taken on that discrete set. Considering the phaseless reconstruction of signals in a shift-invariant space, we introduce a local complement property on a set.

Definition 3.3. *We say that the shift-invariant space $V(\phi)$ has local complement property on a set A if for any $A' \subset A$, there does not exist $f, g \in V(\phi)$ such that $f, g \not\equiv 0$ on A , but $f(x) = 0$ for all $x \in A'$ and $g(y) = 0$ for all $y \in A \setminus A'$.*

The local complement property on the Euclidean space \mathbb{R}^d is the complement property in [20] for ideal sampling functionals on $V(\phi)$, cf. the complement property for frames in Hilbert/Banach spaces ([3, 7, 9, 12]). Local complement property is closely related to local phase retrievability. In fact, following the argument in [20] we have that $V(\phi)$ has the local complement property on A if and only if all signals in $V(\phi)$ is *local phase retrievable* on A , i.e., for any $f, g \in V(\phi)$ satisfying $|g(x)| = |f(x)|, x \in A$, there exists $\delta \in \{-1, 1\}$ such that $g(x) = \delta f(x)$ for all $x \in A$. More discussions on the local complement property is given in Appendix A.

Theorem 3.4. *Let A_1, \dots, A_M be bounded open sets and ϕ be a compactly supported continuous function such that ϕ has local linear independence on $A_m, 1 \leq m \leq M$, and*

$$(3.5) \quad S_k \cap \left(\bigcup_{m=1}^M (A_m + \mathbb{Z}^d) \right) \neq \emptyset$$

for all $k \in \mathbb{Z}^d$ with S_k in (2.5) being nonempty. If the shift-invariant space $V(\phi)$ has local complement property on $A_m, 1 \leq m \leq M$, then there exists a finite set $\Gamma \subset \cup_{m=1}^M A_m$ such that the following statements are equivalent for any signal $f \in V(\phi)$:

- (i) The signal f is determined, up to a sign, from its magnitude measurements on \mathbb{R}^d .
- (ii) The graph \mathcal{G}_f in (2.3) is connected.
- (iii) The signal f is determined, up to a sign, from its phaseless samples $|f(y)|, y \in \Gamma + \mathbb{Z}^d$.

The implication (i) \implies (ii) has been established in Theorem 2.3 and the implication (iii) \implies (i) is obvious. Write $f = \sum_{k \in \mathbb{Z}^d} c(k)\phi(\cdot - k)$. To prove (ii) \implies (iii), we first determine $c(k), k \in K_{A_m} + l$, up to a sign $\epsilon_{m,l} \in \{-1, 1\}$, from phaseless samples $|f(\gamma + l)|, \gamma \in \Gamma$, and then we use the connectivity of the graph \mathcal{G}_f to adjust phases $\epsilon_{m,l} \in \{-1, 1\}, 1 \leq m \leq M, l \in \mathbb{Z}^d$. Finally we sew those pieces together to recover amplitudes $c(k), k \in \mathbb{Z}^d$, and the signal f . The detailed argument will be given in Section 6.3. Comparing with the proof of Theorem 3.1, we remark that our proof of Theorem 3.4 is constructive and a reconstruction algorithm can be developed.

For the case that the generator ϕ has local linear independence on any open set, we can find open sets $A_m, 1 \leq m \leq M$, such that (3.5) holds and $V(\phi)$ has local complement property on $A_m, 1 \leq m \leq M$, see Proposition A.6. Then from Theorem 3.4 we obtain the following corollary, cf. Theorem 2.5 and Corollaries 2.8 and 2.9.

Corollary 3.5. *Let ϕ be a compactly supported continuous function such that ϕ has local linear independence on any open set. Then there exists a finite set Γ such that any nonseparable signal is determined, up to a sign, from its phaseless samples taken on the set $\Gamma + \mathbb{Z}^d$ with finite sampling density.*

Take $\mathbf{N} = (N_1, \dots, N_d)^T$ with $N_i \geq 2, 1 \leq i \leq d$, and let B_{N_i} be the B-spline of order N_i ([19, 45, 46]). Define the box spline function of tensor-product type

$$(3.6) \quad B_{\mathbf{N}}(x) := B_{N_1}(x_1) \times \dots \times B_{N_d}(x_d), \quad x = (x_1, \dots, x_d)^T \in \mathbb{R}^d.$$

As the restriction of a signal in $V(B_{\mathbf{N}})$ on $(0, 1)^d$ is a polynomial of finite degree, the space $V(B_{\mathbf{N}})$ has the local complement property on $(0, 1)^d$. Applying Theorem 3.4 with $M = 1$ and $A_1 = (0, 1)^d$ leads to the following result for tensor-product splines, which is given in [20] for $d = 1$.

Corollary 3.6. *Let X_i contain $2N_i - 1$ distinct points in $(0, 1), 1 \leq i \leq d$. Then any nonseparable signal $f \in V(B_{\mathbf{N}})$ can be reconstructed, up to a sign, from its phaseless samples on the set $X_1 \times \dots \times X_d + \mathbb{Z}^d$ with sampling density $\prod_{i=1}^d (2N_i - 1)$.*

The detailed proof of the above corollary is given in Section 6.4.

In the proof of Theorem 3.4 given in Section 6.3, the discrete sampling set Γ is chosen to be the union of $\Gamma_m \subset A_m, 1 \leq m \leq M$,

$$(3.7) \quad \Gamma = \cup_{m=1}^M \Gamma_m,$$

so that outer products $\Phi_{A_m}(\gamma)(\Phi_{A_m}(\gamma))^T, \gamma \in \Gamma_m$, is a basis (or a spanning set) of the linear space W_{A_m} . Therefore we have the following result from Theorem 3.4.

Corollary 3.7. *Let ϕ and $A_m, 1 \leq m \leq M$, be as in Theorem 3.4. Then any nonseparable signal $f \in V(\phi)$ can be reconstructed from its phaseless samples on a shift-invariant set $\Gamma + \mathbb{Z}^d$ with sampling density*

$$D(\Gamma + \mathbb{Z}^d) \leq \sum_{m=1}^M \dim W_{A_m} \leq \frac{1}{2} \sum_{m=1}^M \#K_{A_m} (\#K_{A_m} + 1).$$

The discrete set $\Gamma + \mathbb{Z}^d$ chosen in Corollary 3.7 may have larger sampling density than $\dim W_{(0,1)^d}$ in Corollary 3.2. Based on the constructive proof in Theorem 3.4, a robust reconstruction algorithm from phaseless samples taken on the discrete set is developed in Section 5. However, we have difficulty to find a reconstruction algorithm from the phaseless samples taken on the set given in Corollary 3.2.

Definition 3.8. We say that $\mathcal{M} = \{a_m \in \mathbb{R}^d, 1 \leq m \leq M\}$ is a *phase retrievable frame* for \mathbb{R}^d if any vector $x \in \mathbb{R}^d$ is determined, up to a sign, from its measurements $|\langle x, a_m \rangle|, a_m \in \mathcal{M}$, and a *minimal phase retrieval frame* for \mathbb{R}^d if any true subset of \mathcal{M} is not a phase retrievable frame.

After careful examination on the proof of Theorem 3.4, we can select a subset Γ' of Γ such that all nonseparable signals f can be reconstructed from its phaseless samples taken on $\Gamma' + \mathbb{Z}^d$ in a robust manner.

Theorem 3.9. *Let $A_m, 1 \leq m \leq M$, and ϕ be as in Theorem 3.4. Assume that there exist $\Gamma'_m \subset A_m$ such that $\Phi_{A_m}(\gamma), \gamma \in \Gamma'_m$, is a phase retrievable frame for $\mathbb{R}^{\#K_{A_m}}$. Then any nonseparable signal $f \in V(\phi)$ is determined, up to a sign, from its phaseless samples $|f(y)|, y \in \Gamma' + \mathbb{Z}^d$, where*

$$(3.8) \quad \Gamma' = \cup_{m=1}^M \Gamma'_m.$$

In Theorem 3.9, the requirement on the sampling set is a bit weaker than the one in Theorem 3.4, as for the sampling set $\Gamma = \cup_{m=1}^M \Gamma_m$ in (3.7), $\Phi_{A_m}(\gamma), \gamma \in \Gamma_m$, is a phase retrievable frame for $\mathbb{R}^{\#K_{A_m}}$, cf. Theorem A.4. We remark that the phase retrieval frame property for $\Phi_A(\gamma), \gamma \in \Gamma$, may not imply that their out products $\Phi_A(\gamma)(\Phi_A(\gamma))^T, \gamma \in \Gamma$, form a basis (or a spanning set) of W_A in (3.3), as shown in the following example.

Example 3.10. Let

$$\phi_1(x) = \begin{cases} x^3/2 & \text{if } 0 \leq x < 1 \\ -x^3 + 3x^2 - 2x + 1/2 & \text{if } 1 \leq x < 2 \\ x^3/2 - 3x^2 + 5x - 3/2 & \text{if } 2 \leq x < 3 \\ 0 & \text{otherwise,} \end{cases}$$

and set $\Phi_1(x) = (\phi_1(x), \phi_1(x+1), \phi_1(x+2))^T, 0 \leq x < 1$. Then

$$\Phi_1(x) = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} x + \frac{1}{2} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} x^3,$$

and

$$\begin{aligned} \Phi_1(x)\Phi_1(x)^T &= \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} x^2 \\ &+ \frac{1}{4} \begin{pmatrix} 0 & 1 & 1 \\ 1 & -4 & -1 \\ 1 & -1 & 2 \end{pmatrix} x^3 + \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ 1 & -4 & 3 \\ -1 & 3 & -2 \end{pmatrix} x^4 + \frac{1}{4} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} x^6. \end{aligned}$$

Therefore the space spanned by $\Phi_1(x), 0 < x < 1$, is \mathbb{R}^3 , and the space $W_{(0,1)}$ spanned by $\Phi_1(x)\Phi_1(x)^T, 0 < x < 1$, is the 6-dimensional linear space of symmetric matrices of size 3×3 . On the other hand, any 3×3 square submatrices of

$$\left(\Phi_1(0) \ \Phi_1\left(\frac{1}{5}\right) \ \Phi_1\left(\frac{2}{5}\right) \ \Phi_1\left(\frac{3}{5}\right) \ \Phi_1\left(\frac{4}{5}\right) \right) = \frac{1}{250} \begin{pmatrix} 0 & 1 & 8 & 27 & 64 \\ 125 & 173 & 209 & 221 & 197 \\ 125 & 76 & 33 & 2 & -11 \end{pmatrix}$$

is nonsingular, which implies that $\Phi_1(m/5), 0 \leq m \leq 4$, forms a phase retrieval frame for \mathbb{R}^3 , but their out products do not form a spanning set of the 6-dimensional space $W_{(0,1)}$.

The problem how to pare down a phase retrieval frame to a minimal phase retrieval frame will be discussed in our future work. Using the pare-down technique, we may find a discrete set X with smaller sampling density such that nonseparable signals in the shift-invariant space can be reconstructed from their phaseless samples taken on X .

4. STABILITY OF PHASELESS SAMPLING AND RECONSTRUCTION

Stability is of paramount importance in the phaseless sampling and reconstruction problem. Consider the scenario that phaseless samples of a signal

$$(4.1) \quad f = \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k) \in V(\phi)$$

taken on a shift-invariant set $\Gamma + \mathbb{Z}^d$ are corrupted by the additive noise,

$$(4.2) \quad z_\epsilon(y) = |f(y)| + \epsilon(y), \ y \in \Gamma + \mathbb{Z}^d,$$

where $\epsilon = (\epsilon(y))_{y \in \Gamma + \mathbb{Z}^d}$ has the bounded noise level $\|\epsilon\|_\infty = \max_{y \in \Gamma + \mathbb{Z}^d} |\epsilon(y)|$, and $\Gamma = \cup_{m=1}^M \Gamma_m$ is either as in (3.7) or in (3.8). In this section, we construct an approximation

$$(4.3) \quad f_\epsilon = \sum_{k \in \mathbb{Z}^d} c_\epsilon(k) \phi(\cdot - k) \in V(\phi),$$

up to a sign, to the original signal f in (4.1) when the noisy phaseless samples (4.2) are available only.

Let

$$(4.4) \quad \Omega_m = \{k \in \mathbb{Z}^d : \phi(\gamma - k) \neq 0 \text{ for some } \gamma \in \Gamma_m\}, \quad 1 \leq m \leq M,$$

and define the hard threshold function $H_\eta, \eta \geq 0$, by

$$H_\eta(t) = \begin{cases} t & \text{if } |t| \geq \eta \\ 0 & \text{if } |t| < \eta. \end{cases}$$

Based on the constructive proofs of Theorems 3.4 and 3.9, we propose the following four-step approach with its implementation discussed in Section 5.

1. Select a phase adjustment threshold $M_0 \geq 0$ and an amplitude threshold $\eta = \sqrt{M_0}$.

2. For $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$, let

$$(4.5) \quad c_{\epsilon, l; m} = (c_{\epsilon, l; m}(k))_{k \in \mathbb{Z}^d}$$

take zero components except that $c_{\epsilon, l; m}(k), k \in l + \Omega_m$, are solutions of the local minimization problem

$$(4.6) \quad \min_{c(k), k \in l + \Omega_m} \sum_{\gamma \in \Gamma_m} \left| \sum_{k \in l + \Omega_m} c(k) \phi(\gamma + l - k) \right| - z_\epsilon(\gamma + l) \Big|^2.$$

3. Adjust phases of $c_{\epsilon, l; m}$ appropriately so that the resulting vectors $\delta_{l, m} c_{\epsilon, l; m}$ with $\delta_{l, m} \in \{-1, 1\}$ satisfy

$$(4.7) \quad \langle \delta_{l, m} c_{\epsilon, l; m}, \delta_{l', m'} c_{\epsilon, l'; m'} \rangle \geq -M_0$$

for all $l, l' \in \mathbb{Z}^d$ and $1 \leq m, m' \leq M$.

4. Sew vectors $\delta_{l, m} c_{\epsilon, l; m}, l \in \mathbb{Z}^d, 1 \leq m \leq M$, together to obtain

$$(4.8) \quad d_\epsilon(k) = \frac{\sum_{m=1}^M \sum_{l \in \mathbb{Z}^d} \delta_{l, m} c_{\epsilon, l; m}(k)}{\sum_{m=1}^M \sum_{l \in \mathbb{Z}^d} \chi_{l + \Omega_m}(k)}, \quad k \in \mathbb{Z}^d.$$

5. Threshold the vector $d_\epsilon = (d_\epsilon(k))_{k \in \mathbb{Z}^d}$,

$$(4.9) \quad c_\epsilon(k) = H_\eta(d_\epsilon(k)), \quad k \in \mathbb{Z}^d$$

to define the approximation f_ϵ in (4.3).

In the next theorem, we show that the above approach provides a sub-optimal approximation to the original signal in a noisy phaseless sampling environment.

Theorem 4.1. *Let A_1, \dots, A_M be bounded open sets satisfying (3.5), ϕ be a compactly supported continuous function such that ϕ has local linear independence on $A_m, 1 \leq m \leq M$, and $\Gamma_m \subset A_m$ be so chosen that $\Phi_{A_m}(\gamma), \gamma \in \Gamma_m$, is a phase retrievable frame for $\mathbb{R}^{K_{A_m}}$. Assume that the graph $\mathcal{G}_f = (V_f, E_f)$*

of the original signal $f = \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k)$ is connected and

$$(4.10) \quad F_0 := \inf_{k \in V_f} |c(k)|^2 > 0.$$

Set $\Gamma = \cup_{m=1}^M \Gamma_m$ and

$$(4.11) \quad \|\Phi^{-1}\|_2 = \sup_{\Theta_m \subset \Gamma_m, 1 \leq m \leq M} \left(\min_{\|d\|_2=1} \left(\sup_{\|d\|_2=1} \|\Phi_{\Theta_m} d\|_2^{-1}, \sup_{\|d\|_2=1} \|\Phi_{\Gamma_m \setminus \Theta_m} d\|_2^{-1} \right) \right)^{-1},$$

where $\Phi_{\Theta_m} = (\phi(\gamma - k))_{\gamma \in \Theta_m, k \in \Omega_m}$ for $\Theta_m \subset \Gamma_m$. If the phase adjustment threshold constant $M_0 \geq 0$ and the noise level $\|\epsilon\|_\infty := \sup_{y \in \Gamma + \mathbb{Z}^d} |\epsilon(y)|$ satisfy

$$(4.12) \quad M_0 \leq \frac{2F_0}{9},$$

and

$$(4.13) \quad 8\#\Gamma \|\Phi^{-1}\|_2^2 \|\epsilon\|_\infty^2 \leq M_0,$$

then the signal $f_\epsilon = \sum_{k \in \mathbb{Z}^d} c_\epsilon(k) \phi(\cdot - k) \in V(\phi)$ reconstructed from the proposed approach (4.5)–(4.9) satisfies

$$(4.14) \quad |c_\epsilon(k) - \delta c(k)| \leq 2\sqrt{\#\Gamma} \|\Phi^{-1}\|_2 \|\epsilon\|_\infty, \quad k \in V_f$$

and

$$(4.15) \quad c_\epsilon(k) = c(k) = 0, \quad k \notin V_f,$$

where $\delta \in \{-1, 1\}$.

By Theorem 4.1, the reconstructed signal f_ϵ in (4.3) provides a suboptimal approximation, up to a sign, to the original signal f in (4.1),

$$(4.16) \quad \|f_\epsilon - \delta f\|_\infty \leq 2\sqrt{\#\Gamma} \|\Phi^{-1}\|_2 \left(\sup_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |\phi(x - k)| \right) \|\epsilon\|_\infty$$

and

$$(4.17) \quad \sup_{y \in \Gamma + \mathbb{Z}^d} ||f_\epsilon(y)| - |f(y)|| \leq 2\sqrt{\#\Gamma} \|\Phi^{-1}\|_2 \left(\sup_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |\phi(x - k)| \right) \|\epsilon\|_\infty.$$

By (4.10), (4.12), (4.13) and (4.14), a vertex in the graph \mathcal{G}_f is also a vertex of the graph \mathcal{G}_{f_ϵ} associated with the reconstructed signal f_ϵ . This together with (2.3) and (4.15) implies that the graphs \mathcal{G}_f and \mathcal{G}_{f_ϵ} associated with the original signal f and the reconstructed signal f_ϵ are the same, i.e.,

$$\mathcal{G}_f = \mathcal{G}_{f_\epsilon}.$$

The selection of the threshold constant $M_0 \geq 0$ is imperative to find an approximation to the original signal from its phaseless samples. In the noiseless environment, we may take $M_0 = 0$ and the proposed approach leads to a perfect reconstruction, i.e., $f_\epsilon = \pm f$, when f is nonseparable. In practical applications, the noise level is positive and the phase adjustment

threshold constant M_0 needs to be appropriately selected. For instance, we may require that (4.12) and (4.13) are satisfied if we have some prior information about the amplitude vector of the original signal. From the proof of Theorem 4.1 and also the simulations in the next section, it is observed that phases can not be adjusted to satisfy (4.7) if M_0 is far below square of noise level $\|\epsilon\|_\infty$ (for instance, (4.13) is not satisfied), while the phase adjustment (4.7) in the algorithm is not essentially determined and hence the reconstructed signal is not a good approximation of the original signal if M_0 is comparable to the square of minimal magnitude of amplitude vector of the original signal (for instance, (4.12) is not satisfied).

Remark 4.2. By Theorem 4.1, there is no resonance phenomenon in the sense that

$$(4.18) \quad \inf_{\delta \in \{-1, 1\}} \|f_\epsilon - \delta f\|_\infty \leq C \|\epsilon\|_\infty$$

if the noise level is far below the minimal magnitude of amplitude vector of the original signal, i.e.,

$$(4.19) \quad \|\epsilon\|_\infty \leq C_0 \inf_{k \in V_f} |c(k)|$$

for some sufficiently small constant C_0 . The phaseless sampling and reconstruction problem is ill-posed if the noise level is high. For instance, the estimate (4.18) is not satisfied for the nonseparable spline signal of order 2,

$$f_\alpha(x) = B_2(x) + \alpha B_2(x-1) + B_2(x-2) \in V(B_2),$$

if $\|\epsilon\|_\infty \geq 2\alpha/(1+\alpha)$, where $\alpha \in (0, 1)$ is sufficiently small. The reasons are that the signal $\tilde{f}_\alpha(x) = B_2(x) + \alpha B_2(x-1) - B_2(x-2) \in V(B_2)$ satisfy

$$\min_{\delta \in \{-1, 1\}} \|f_\alpha - \delta \tilde{f}_\alpha\|_\infty = 2 \quad \text{and} \quad \| |f_\alpha| - |\tilde{f}_\alpha| \|_\infty = \frac{2\alpha}{1+\alpha}.$$

5. RECONSTRUCTION ALGORITHM AND NUMERICAL SIMULATIONS

Consider the scenario that phaseless samples of a signal $f = \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k) \in V(\phi)$ taken on a finite set $\Gamma + K \subset \Gamma + \mathbb{Z}^d$ are corrupted by the additive noise,

$$(5.1) \quad z_\epsilon(y) = |f(y)| + \epsilon(y), \quad y \in \Gamma + K,$$

where $\epsilon(y) \in [-\epsilon, \epsilon]$, $y \in \Gamma + K$, for some $\epsilon \geq 0$, and $\Gamma = \cup_{m=1}^M \Gamma_m$ is either as in (3.7) or in (3.8). Define

$$(5.2) \quad f_K = \sum_{k \in \tilde{K}} c(k) \phi(\cdot - k),$$

where $\tilde{K} = \cup_{l \in K} \cup_{m=1}^M (l + \Omega_m)$ and Ω_m , $1 \leq m \leq M$, are as in (4.4). Then the noisy data $z_\epsilon(y)$, $y \in \Gamma + K$, in (5.1) is

$$(5.3) \quad z_\epsilon(y) = |f_K(y)| + \epsilon(y) \geq 0, \quad y \in \Gamma + K.$$

Based on (5.3) and the four-step approach in Section 4, we propose an algorithm to find an approximation f_ϵ of the form

$$(5.4) \quad f_\epsilon = \sum_{k \in \tilde{K}} c_\epsilon(k) \phi(\cdot - k) \in V(\phi),$$

up to a sign, to the original signal f in (5.2) when the noisy phaseless samples (5.1) are available only. The algorithm contains four parts: **minimization**, **adjusting phases**, **sewing** and **thresholding**, and we call it the MAPSET algorithm.

Algorithm 1 MAPSET Algorithm

Input: finite set $K \subset \mathbb{Z}^d$; sampling set $\Gamma = \cup_{m=1}^M \Gamma_m$ either in (3.7) or in (3.8); noisy phaseless sampling data $(z_\epsilon(y))_{y \in \Gamma + K}$; index set $\tilde{K} = \cup_{l \in K} \cup_{m=1}^M (l + \Omega_m) \subset \mathbb{Z}^d$; and the phase adjustment threshold constant M_0 .

Initials: Start from zero vectors $c_{\epsilon,l;m} = (c_{\epsilon,l;m}(k))_{k \in \tilde{K}}, l \in K, 1 \leq m \leq M$.

Instructions:

1) Local minimization: For $l \in K$ and $1 \leq m \leq M$, replace $c_{\epsilon,l;m}(k), k \in l + \Omega_m$, by a solution of the local minimization problem

$$\min_{c(k), k \in l + \Omega_m} \sum_{\gamma \in \Gamma_m} \left| \sum_{k \in l + \Omega_m} c(k) \phi(\gamma + l - k) - z_\epsilon(\gamma + l) \right|^2.$$

2) Phase adjustment: For $l \in K$ and $1 \leq m \leq M$, multiply $c_{\epsilon,l;m}$ by $\delta_{l,m} \in \{-1, 1\}$ so that $\langle \delta_{l,m} c_{\epsilon,l;m}, \delta_{l',m'} c_{\epsilon,l',m'} \rangle \geq -M_0$ for all $l, l' \in K$ and $1 \leq m, m' \leq M$.

3) Sewing local approximations:

$$d_\epsilon(k) = \frac{\sum_{m=1}^M \sum_{l \in K} \delta_{l,m} c_{\epsilon,l;m}(k)}{\sum_{m=1}^M \sum_{l \in K} \chi_{l + \Omega_m}(k)}, \quad k \in \tilde{K}.$$

4) Hard thresholding:

$$c_\epsilon(k) = \begin{cases} d_\epsilon(k) & \text{if } |d_\epsilon(k)| \geq \sqrt{M_0} \\ 0 & \text{else} \end{cases}, \quad k \in \tilde{K}.$$

Output: Amplitude vector $(c_\epsilon(k))_{k \in \tilde{K}}$, and the reconstructed signal $f_\epsilon = \sum_{k \in \tilde{K}} c_\epsilon(k) \phi(\cdot - k)$.

In this section, we also demonstrate the performance of the proposed MAPSET algorithm on reconstructing box spline signals from their noisy phaseless samples on a discrete set.

5.1. Nonseparable spline signals of tensor-product type. Let $B_{(3,3)}$ be the tensor product of one-dimensional quadratic spline B_3 , see (3.6). For $A = (0, 1)^2$ and $\phi = B_{(3,3)}$, the vector-valued function Φ_A in (2.6) and the set K_A in (2.7) becomes

$$(5.5) \quad \Phi_{(0,1)^2}(s, t) = (b_i(s)b_j(t))_{(i,j) \in K_{(0,1)^2}}, \quad (s, t) \in (0, 1)^2$$

and $K_{(0,1)^2} = \{(i, j) : -2 \leq i, j \leq 0\}$ respectively, where $b_0(s) = s^2/2$, $b_{-1}(s) = (-2s^2 + 2s + 1)/2$ and $b_{-2}(s) = (1-s)^2/2$, $0 \leq s \leq 1$. One may verify that the space spanned by the outer products of $\Phi_{(0,1)^2}(s, t)$, $(s, t) \in (0, 1)^2$, has dimension 25, and the set

$$(5.6) \quad \Gamma_0 = \{(i, j)/6, 1 \leq i, j \leq 5\} \subset (0, 1)^2$$

with cardinality 25 satisfies (3.7), see Figure 1. For the above uniformly distributed set Γ_0 , the corresponding $\|\Phi^{-1}\|_2$ in (4.11) is 2.7962×10^3 .

As $\Phi_{(0,1)^2}(s, t)$, $(s, t) \in (0, 1)^2$, is a 9-dimensional vector-valued polynomial about $s^m t^n$, $0 \leq m, n \leq 2$, the shift-invariant space generated by $B_{(3,3)}$ has local complement property on $(0, 1)^2$. Observe that the matrix $(\Phi_{(0,1)^2}(s_i, t_i))_{1 \leq i \leq 9}$ has full rank 9 for almost all $(s_i, t_i) \in (0, 1)^2$, $1 \leq i \leq 9$. Hence $(\Phi_{(0,1)^2}(s_i, t_i))_{1 \leq i \leq 17}$ is a phase retrieval frame for almost all $(s_i, t_i) \in (0, 1)^2$, $1 \leq i \leq 17$, but the corresponding $\|\Phi^{-1}\|_2$ in (4.11) are relatively large from our numerical calculation. So we use a randomly distributed set $\Gamma_1 \subset (0, 1)^2$ with cardinality 19 in our simulations, see Figure 1. The above set satisfies (3.8) and the corresponding $\|\Phi^{-1}\|_2$ in (4.11) is 3.2995×10^4 .

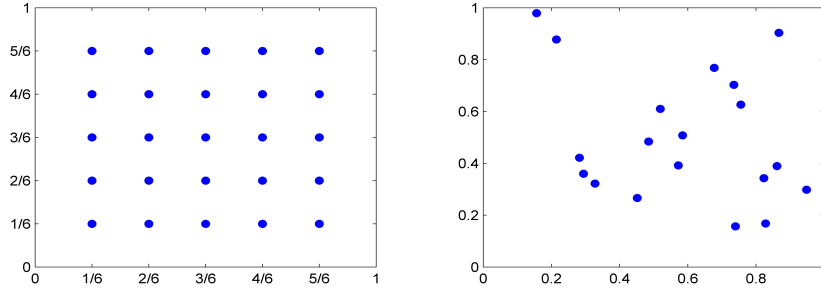


FIGURE 1. Plotted on the left is a uniformly distributed set Γ_0 satisfying (3.7), while on the right is a randomly distributed set Γ_1 satisfying (3.8). The corresponding $\|\Phi^{-1}\|_2$ in (4.11) to the above sets are 2.7962×10^3 (left) and 3.2995×10^4 (right), respectively.

In our simulations, the available data $z_\epsilon(y) = |f(y)| + \epsilon(y) \geq 0$, $y \in \Gamma + K$, are noisy phaseless samples of a spline signal

$$(5.7) \quad f(s, t) = \sum_{0 \leq m \leq K_1, 0 \leq n \leq K_2} c(m, n) B_{(3,3)}(s - m, t - n),$$

taken on $\Gamma + K$, where $K = [0, K_1] \times [0, K_2]$ for some positive integers $K_1, K_2 \geq 1$, Γ is either the uniform set Γ_0 or the random set Γ_1 in Figure 1, amplitudes of the signal f ,

$$(5.8) \quad c(m, n) \in [-1, 1] \setminus [-0.1, 0.1], \quad 0 \leq m \leq K_1, 0 \leq n \leq K_2,$$

are randomly chosen, and the additive noises $\epsilon(y) \in [-\epsilon, \epsilon]$, $y \in \Gamma + K$, with noise level $\epsilon \geq 0$ are randomly selected. Denote the signal reconstructed by the proposed MAPSET algorithm with phase adjustment thresholding constant $M_0 = 0.01$, cf. (4.12) with $F_0 = 0.01$, by

$$(5.9) \quad f_\epsilon(s, t) = \sum_{-2 \leq m \leq K_1, -2 \leq n \leq K_2} c_\epsilon(m, n) B_{(3,3)}(s - m, t - n).$$

Define maximal amplitude error of the MAPSET algorithm by

$$(5.10) \quad e(\epsilon) := \min_{\delta \in \{-1, 1\}} \max_{-2 \leq m \leq K_1, -2 \leq n \leq K_2} |c_\epsilon(m, n) - \delta c(m, n)|.$$

As the original spline signal f in (5.7) is nonseparable, the conclusions (4.14) and (4.15) guarantee that the reconstruction signal f_ϵ provides an approximation, up to a sign, to the original signal f if $\|\Phi^{-1}\|_2 \epsilon$ is much smaller than a multiple of $\sqrt{M_0}$, where M_0 is the phase adjustment thresholding constant. Our numerical simulation indicates that the MAPSET algorithm saves phases successfully in 100 trials and the maximal amplitude error $e(\epsilon)$ in (5.10) is about $O(\epsilon)$, provided that $\epsilon \leq 2 \times 10^{-3}$ for $\Gamma = \Gamma_0$ and $\epsilon \leq 7 \times 10^{-4}$ for $\Gamma = \Gamma_1$, where $\sqrt{M_0}/\|\Phi^{-1}\|_2$ are 3.5763×10^{-5} and 3.0307×10^{-6} respectively. Presented in Figure 2 are a spline signal f in (5.7) with $K_1 = K_2 = 9$ and the difference between the original signal f and the reconstructed signal f_ϵ via the MAPSET algorithm with noise level $\epsilon = 10^{-4}$.

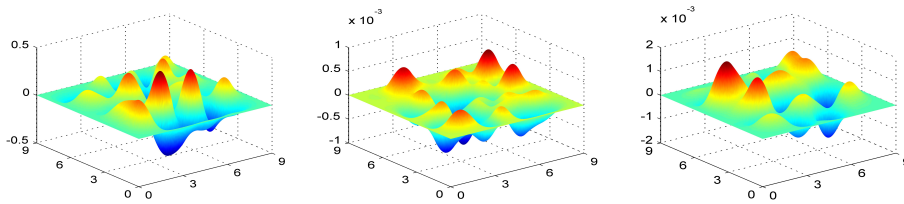


FIGURE 2. Plotted on the left is a spline signal in (5.7) with $K_1 = K_2 = 9$. In the middle and on the right are the difference between the above signal f and the signal f_ϵ reconstructed by the MAPSET algorithm with noise level $\epsilon = 10^{-4}$ and the sampling set Γ being Γ_0 and Γ_1 in Figure 1, respectively. The maximal amplitude errors $e(\epsilon)$ in (5.10) is 0.0014 (middle) and 0.0030 (right), and the reconstruction error $\min_{\delta \in \{-1, 1\}} \|f_\epsilon - \delta f\|_\infty$ is 7.2567×10^{-4} (middle) and 0.0015 (right), respectively.

The signal f_ϵ reconstructed from the MAPSET algorithm may not provide a good approximation, up to a sign, to the original signal f if the noise level ϵ is larger than a multiple of $\sqrt{M_0}/\|\Phi^{-1}\|_2$, cf. (4.13) in Theorem 4.1. Our numerical simulations indicate that the MAPSET algorithm sometimes fails to save the phase of the original signal f when $\epsilon \geq 3 \times 10^{-3}$ for $\Gamma = \Gamma_0$ and $\epsilon \geq 8 \times 10^{-4}$ for $\Gamma = \Gamma_1$.

5.2. Nonseparable spline signals of non-tensor product type.

Let M_{Ξ_Z} be the box spline function in (1.2) with $\Xi_Z = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, see [19]. Unlike the spline function $B_{(3,3)}$ of tensor-product type, the shift-invariant space spanned by M_{Ξ_Z} does not have the local complement property on $(0, 1)^2$, cf. Section 5.1. Set $A_U := \{(s, t) : 0 < s < t < 1\}$ and $A_L := \{(s, t) : 0 < t < s < 1\}$. Then the triangle regions A_U and A_L satisfy (3.5), and the shift-invariant space spanned by M_{Ξ_Z} has local complement property on A_U and on A_L .

For $A = A_U$ and $\phi = M_{\Xi_Z}$, the function $\Phi_{A_U}(s, t)$ in (2.6) is a 5-dimensional vector-valued polynomial about $s^2, (t-s)^2, s, t-s, 1$, and the set K_{A_U} in (2.7) is $\{(0, 0), (-1, 0), (-2, 0), (-1, -1), (-2, -1)\}$. Hence the space spanned by the outer products of $\Phi_{A_U}(s, t)$ has dimension 13, and we can select a set $\Gamma_{2,U} \subset A_U$ with cardinality 13 to satisfy (3.7), see Figure 3. Similarly, for the lower triangle region A_L , a sampling set $\Gamma_{2,L}$ with cardinality 13 can be chosen to satisfy (3.7). For our simulations, we use

$$\Gamma_2 = \Gamma_{2,U} \cup \Gamma_{2,L}$$

as the sampling set contained in $A_U \cup A_L \subset (0, 1)^2$, see Figure 3. For the above set Γ_2 , the corresponding $\|\Phi^{-1}\|_2$ in (4.11) is 87.9420.

Recall $\Phi_{A_U}(s, t)$ is a vector-valued polynomial about $s^2, (t-s)^2, s, t-s$ and 1. Then the matrix $(\Phi_{A_U}(s_i, t_i))_{1 \leq i \leq 5}$ has full rank 5 for almost all $(s_i, t_i) \in A_U, 1 \leq i \leq 5$, and $(\Phi_{A_U}(s_i, t_i))_{1 \leq i \leq 9}$ is a phase retrieval frame for almost all $(s_i, t_i) \in A_U, 1 \leq i \leq 9$. So we can use randomly distributed sets $\Gamma_{3,U} \subset A_U$ and $\Gamma_{3,L} \subset A_L$ with cardinality 9 that satisfy (3.8), see Figure 3. Set

$$\Gamma_3 = \Gamma_{3,U} \cup \Gamma_{3,L}.$$

For the above set Γ_3 , the corresponding $\|\Phi^{-1}\|_2$ in (4.11) is 761.2227.

In our simulations, the available data $z_\epsilon(y) = |f(y)| + \epsilon(y) \geq 0, y \in \Gamma + K$, are noisy phaseless samples of a spline signal

$$(5.11) \quad f(s, t) = \sum_{0 \leq m \leq K_1, 0 \leq n \leq K_2} c(m, n) M_{\Xi_Z}(s - m, t - n),$$

taken on $\Gamma + K$, where $K = [0, K_1] \times [0, K_2]$ for some $1 \leq K_1, K_2 \in \mathbb{Z}$, Γ is either Γ_2 or Γ_3 in Figure 3, amplitudes of the signal f are as in (5.8), and the additive noises $\epsilon(y) \in [-\epsilon, \epsilon], y \in \Gamma + K$, with noise level $\epsilon \geq 0$

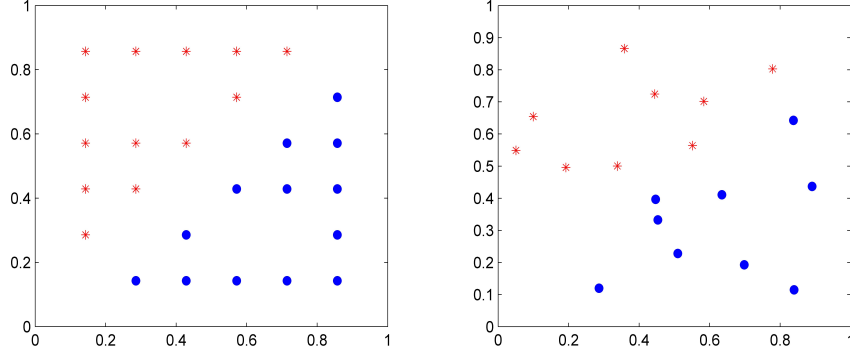


FIGURE 3. Plotted on the left are the sampling sets $\Gamma_{2,U} \subset A_U$ (in the red star) and $\Gamma_{2,L} \subset A_L$ (in the blue dot). Plotted on the right are the random sets $\Gamma_{3,U} \subset A_U$ (in the red star) and $\Gamma_{3,L} \subset A_L$ (in the blue dot) that have cardinality 9. The corresponding $\|\Phi^{-1}\|_2$ in (4.11) to the above sets is 87.9420 (left) and 761.2227 (right) respectively.

are randomly selected. Denote the signal reconstructed by the proposed MAPSET algorithm with thresholding constant $M_0 = 0.01$ by

$$(5.12) \quad f_\epsilon(s, t) = \sum_{-2 \leq m \leq K_1, -1 \leq n \leq K_2} c_\epsilon(m, n) M_{\Xi_Z}(s - m, t - n).$$

As in Section 5.1, the reconstruction signal f_ϵ provides an approximation, up to a sign, to the original signal f . Our numerical simulation indicates that the MAPSET algorithm saves phases in 1000 trials and the reconstruction error $e(\epsilon)$ is about $O(\epsilon)$, provided that $\epsilon \leq 8 \times 10^{-3}$ for $\Gamma = \Gamma_2$ and $\epsilon \leq 4 \times 10^{-3}$ for $\Gamma = \Gamma_3$, where $\sqrt{M_0}/\|\Phi^{-1}\|_2$ are 0.0011 and 1.3137×10^{-4} respectively. Presented in Figure 4 are a spline signal f in (5.11) with $(K_1, K_2) = (9, 8)$, and the difference between the original signal f and the reconstructed signal f_ϵ via the MAPSET algorithm with noise level $\epsilon = 10^{-4}$.

As in Section 5.1, the MAPSET algorithm may not yield a good approximation to the original signal if the noisy level ϵ is not sufficient small. Our numerical results indicate that the MAPSET algorithm sometimes fails to save the phase of the original signal f when $\epsilon \geq 9 \times 10^{-3}$ for $\Gamma = \Gamma_2$ and $\epsilon \geq 5 \times 10^{-3}$ for $\Gamma = \Gamma_3$.

6. PROOFS

In this section, we include the proofs of Theorems 2.3, 3.1, 3.4, 3.9, 2.5, 4.1 and Corollary 3.6.

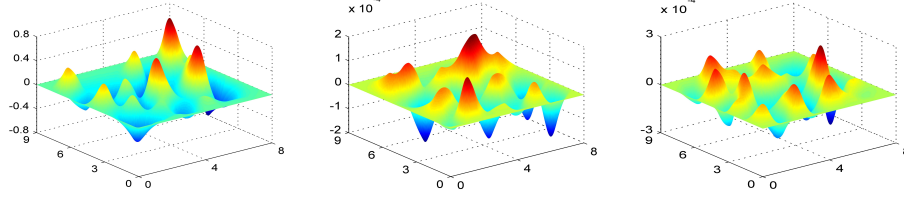


FIGURE 4. Plotted on the left is a nonseparable signal of the form (5.11), where $K = [0, 9] \times [0, 8]$, and in the middle and on the right are the difference between the above signal f and the signal f_ϵ reconstructed by the MAPSET algorithm with noise level $\epsilon = 10^{-4}$ and the sampling set Γ being Γ_2 and Γ_3 in Figure 3, respectively. The maximal amplitude error $e(\epsilon)$ in (5.10) is 2.4922×10^{-4} (middle) and 3.8975×10^{-4} (right). The reconstruction error $\min_{\delta \in \{-1, 1\}} \|f_\epsilon - \delta f\|_\infty$ is 1.9660×10^{-4} (middle) and 2.9216×10^{-4} (right).

6.1. Proof of Theorem 2.3. Suppose, on the contrary, that \mathcal{G}_f is disconnected. Let W be the set of vertices in a connected component of the graph \mathcal{G}_f . Then $W \neq \emptyset$, $V_f \setminus W \neq \emptyset$, and there are no edges between vertices in W and $V_f \setminus W$. Write

$$\begin{aligned}
 f &= \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k) = \sum_{k \in V_f} c(k) \phi(\cdot - k) \\
 (6.1) \quad &= \sum_{k \in W} c(k) \phi(\cdot - k) + \sum_{k \in V_f \setminus W} c(k) \phi(\cdot - k) =: f_1 + f_2
 \end{aligned}$$

where $c(k) \in \mathbb{R}$, $k \in \mathbb{Z}^d$. From the global linear independence on ϕ and nontriviality of the sets W and $V_f \setminus W$, we obtain

$$(6.2) \quad f_1 \neq 0 \quad \text{and} \quad f_2 \neq 0.$$

Combining (6.1) and (6.2) with nonseparability of the signal f , we obtain that $f_1(x_0)f_2(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^d$. Then by the global linear independence of ϕ , there exist $k \in W$ and $k' \in V_f \setminus W$ such that $\phi(x_0 - k) \neq 0$ and $\phi(x_0 - k') \neq 0$. Hence (k, k') is an edge between $k \in W$ and $k' \in V_f \setminus W$, which contradicts to the construction of the set W .

6.2. Proof of Theorem 3.1. A linear space V on \mathbb{R}^d is said to be *locally finite-dimensional* if it has finite dimensional restrictions on any bounded open set. The shift-invariant space generated by a compactly supported function is locally finite-dimensional. The reader may refer to [5] and references therein on locally finite-dimensional spaces. In this section, we will prove the following generalization of Theorem 3.1.

Theorem 6.1. *Let V be locally finite-dimensional shift-invariant space of functions on \mathbb{R}^d . Then there exists a finite set $\Gamma \subset (0, 1)^d$ such that any non-separable signal $f \in V$ can be reconstructed, up to a sign, from its phaseless samples on $\Gamma + \mathbb{Z}^d$.*

Proof. Let $A = (0, 1)^d$ and $V|_A$ be the space containing restrictions of all signals in V on A . By the shift-invariance, it suffices to find a set $\Gamma \subset A$ such that

$$(6.3) \quad |f(x)|^2 = \sum_{\gamma \in \Gamma} d_\gamma(x) |f(\gamma)|^2, \quad x \in A$$

hold for all $f \in V$. By the assumption on V , $V|_A$ is a finite-dimensional space. Let $g_n \in V$, $1 \leq n \leq N$, be a basis for $V|_A$, and W be the linear space generated by symmetric matrices

$$G(x) := (g_n(x)g_{n'}(x))_{1 \leq n, n' \leq N}, \quad x \in A.$$

Then there exists a finite set $\Gamma \subset A$ with cardinality at most $N(N+1)/2$ such that $G(\gamma)$, $\gamma \in \Gamma$, is a basis for the space W . This implies that for any $x \in A$ there exist $d_\gamma(x)$, $\gamma \in \Gamma$, such that

$$G(x) = \sum_{\gamma \in \Gamma} d_\gamma(x) G(\gamma), \quad x \in A.$$

For any $f \in V$, we write $f = \sum_{n=1}^N c_n g_n$ on A . Then

$$\begin{aligned} |f(x)|^2 &= \left| \sum_{n=1}^N c_n g_n(x) \right|^2 = \sum_{n, n'=1}^N c_n c_{n'} g_n(x) g_{n'}(x) \\ &= \sum_{n, n'=1}^N c_n c_{n'} \left(\sum_{\gamma \in \Gamma} d_\gamma(x) g_n(\gamma) g_{n'}(\gamma) \right) = \sum_{\gamma \in \Gamma} d_\gamma(x) |f(\gamma)|^2, \quad x \in A. \end{aligned}$$

This proves (6.3) and hence completes the proof. \square

6.3. Proof of Theorem 3.4. The implication (iii) \implies (i) is obvious, while the implication (i) \implies (ii) follows from Theorem 2.3 with its proof given in Section 6.1. Now it remains to prove (ii) \implies (iii). Let Γ_m , $1 \leq m \leq M$, be finite sets constructed in Proposition A.3 with the set A and the space V replaced by A_m and $V(\phi)$ respectively, and set $\Gamma = \cup_{m=1}^M \Gamma_m$.

Let $f, g \in V(\phi)$ satisfy

$$(6.4) \quad |g(y)| = |f(y)| \quad \text{for all } y \in \Gamma + \mathbb{Z}^d.$$

Then it suffices to prove that

$$(6.5) \quad g = \epsilon f$$

for some $\epsilon \in \{-1, 1\}$. By Proposition A.3 and the shift-invariance of the linear space $V(\phi)$,

$$|g(x+l)| = |f(x+l)|, \quad x \in A_m$$

where $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$. This, together shift-invariance of the linear space $V(\phi)$ and local complement property on A_m , $1 \leq m \leq M$, implies the existence of $\epsilon_{l,m} \in \{-1, 1\}$ such that

$$(6.6) \quad g(x) = \epsilon_{l,m} f(x), \quad x \in A_m + l.$$

Write $f = \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k)$ and $g = \sum_{k \in \mathbb{Z}^d} d(k) \phi(\cdot - k) \in V(\phi)$. Then it follows from (6.6) and local linear independence on A_m for the generator ϕ that

$$(6.7) \quad d(k' + l) = \epsilon_{l,m} c(k' + l), \quad k' \in K_{A_m},$$

where K_{A_m} is given in (2.7). Hence the proof of (6.5) reduces to showing

$$(6.8) \quad \epsilon_{l,m} = \epsilon$$

for all $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$ so that $k' + l \in V_f$ for some $k' \in K_{A_m}$.

Recall that $c(k) \neq 0$ for all $k \in V_f$. Then by (6.7) there exist $\delta_k \in \{-1, 1\}$ for all $k \in V_f$ such that

$$(6.9) \quad \epsilon_{l,m} = \delta_k$$

for all $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$ so that $k = k' + l \in V_f$ for some $k' \in K_{A_m}$. By (6.9) and the connectivity of the graph \mathcal{G}_f , the proof of (6.8) reduces further to proving

$$(6.10) \quad \delta_k = \delta_{\tilde{k}}$$

for all edges (k, \tilde{k}) of the graph \mathcal{G}_f .

For an edge (k, \tilde{k}) of the graph \mathcal{G}_f , we have that

$$S := \{x : \phi(x - k) \phi(x - \tilde{k}) \neq 0\} \neq \emptyset.$$

Then there exist $1 \leq m \leq M$ by (2.5) and (3.5) such that $S \cap (A_m + k) \neq \emptyset$. Thus $k, \tilde{k} \in K_{A_m} + k$, which together with (6.7) and (6.9) implies that $\delta_k = \epsilon_{k,m} = \delta_{\tilde{k}}$. Hence (6.10) holds. This completes the proof.

6.4. Proof of Corollary 3.6. As restriction of signals in $V(B_{\mathbf{N}})$ on $(0, 1)^d$ are polynomials of finite degree, the space $V(B_{\mathbf{N}})$ has the complement property on $(0, 1)^d$. Set $\mathbf{n} = (n, \dots, n)$ for $n \in \mathbb{Z}$. It is observed that the function $\Phi_{(0,1)^d}$ in (2.6) is a vector-valued polynomial of degree $\mathbf{N} - \mathbf{1}$, and its outer product $\Phi_{(0,1)}(x) \Phi_{(0,1)}(x)^T$ is a matrix-valued polynomial of degree $2\mathbf{N} - \mathbf{2}$. Therefore $\Phi_{(0,1)}(y) \Phi_{(0,1)}(y)^T, y \in X_1 \times \dots \times X_d$, is a spanning set of the space spanned by $\Phi_{(0,1)}(x) \Phi_{(0,1)}(x)^T, x \in (0, 1)^d$. This together with Theorem 3.4 completes the proof.

6.5. Proof of Theorem 3.9. Let $f = \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k)$ and $g = \sum_{k \in \mathbb{Z}^d} d(k) \phi(\cdot - k)$ satisfy

$$|g(y)| = |f(y)| \quad \text{for all } y \in \Gamma' + \mathbb{Z}^d,$$

where $\Gamma' = \cup_{m=1}^M \Gamma'_m$ is given in (3.8). Take $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$. Then

$$\left| \sum_{k \in K_{A_m} + l} d(k) \phi(\gamma_m + l - k) \right| = \left| \sum_{k \in K_{A_m} + l} c(k) \phi(\gamma_m + l - k) \right| \quad \text{for all } \gamma \in \Gamma'_m.$$

By the assumption on $\Phi_{A_M}(\gamma'), \gamma \in \Gamma'_m, 1 \leq m \leq M$, and the shift-invariance of the linear space $V(\phi)$, there exists $\epsilon_{m,l} \in \{1, -1\}$ such that

$$d(k) = \epsilon_{m,l} c(k), \quad k \in K_{A_m} + l.$$

Following the same argument as the one used in the implication i) \implies iii) in Theorem 3.4, we can find $\epsilon \in \{-1, 1\}$ such that $\epsilon_{l,m} = \epsilon$ for all $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$. This completes the proof.

6.6. Proof of Theorem 2.5. By Proposition A.6, there are open sets A_1, \dots, A_M satisfy the requirements in Theorem 3.4. Then the conclusion in Theorem 2.5 follows from Theorem 3.4.

6.7. Proof of Theorem 4.1. Given $\Gamma \subset \mathbb{R}^d$ and $f = \sum_{k \in \mathbb{Z}^d} c(k) \phi(\cdot - k)$, we define

$$(6.11) \quad \tilde{\mathcal{G}}_{f,\Gamma} = (V_f, E_{f,\Gamma}),$$

where $(k, k') \in E_{f,\Gamma}$ only if $\phi(y - k) \phi(y - k') \neq 0$ for some $y \in \Gamma + \mathbb{Z}^d$. To prove Theorem 4.1, we need a lemma about the graph \mathcal{G}_f .

Lemma 6.2. *Let ϕ , A_m and $\Gamma_m, 1 \leq m \leq M$, be as in Theorem 4.1. Set $\Gamma = \cup_{m=1}^M \Gamma_m$. Then for any $f \in V(\phi)$, the graph \mathcal{G}_f in (2.3) and $\tilde{\mathcal{G}}_{f,\Gamma}$ in (6.11) are the same,*

$$(6.12) \quad \mathcal{G}_f = \tilde{\mathcal{G}}_{f,\Gamma}.$$

Proof. Clearly it suffices to prove that an edge in \mathcal{G}_f is also an edge in $\tilde{\mathcal{G}}_{f,\Gamma}$. Suppose, on the contrary, that there exists an edge $(k, k') \in E_f$ such that

$$(6.13) \quad \phi(y - k) \phi(y - k') = 0 \quad \text{for all } y \in \cup_{m=1}^M \Gamma_m + \mathbb{Z}^d.$$

Define

$$(6.14) \quad A = \{x \in \mathbb{R}^d : \phi(x - k) \phi(x - k') \neq 0\} \neq \emptyset.$$

By (3.5), there exist $l_0 \in \mathbb{Z}^d$ and $1 \leq m_0 \leq M$ such that

$$(6.15) \quad A \cap (A_{m_0} + l_0) \neq \emptyset.$$

Set $g_{\pm}(x) = \phi(x + l_0 - k) \pm \phi(x + l_0 - k')$. Then it follows from (6.13) that

$$|g_{\pm}(\gamma)| = |\phi(\gamma + l_0 - k)| + |\phi(\gamma + l_0 - k')|, \quad \gamma \in \Gamma_{m_0}.$$

By the construction of the set Γ_{m_0} , we get either $g_+ = g_-$ or $g_+ = -g_-$ on A_{m_0} . Therefore either $\phi(x + l_0 - k) \equiv 0$ on A_{m_0} or $\phi(x + l_0 - k') \equiv 0$ on A_{m_0} . This contradicts to the construction of set A in (6.14) and (6.15). \square

Now, we continue the proof of Theorem 4.1.

Proof of Theorem 4.1. Take $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$. For $\gamma \in \Gamma_m$ there exists $\tilde{\delta}_{\gamma,l,m} \in \{-1, 1\}$ such that

$$\begin{aligned}
& \left(\sum_{\gamma \in \Gamma_m} \left| \sum_{k \in l + \Omega_m} (c_{\epsilon,l;m}(k) - \tilde{\delta}_{\gamma,l,m} c(k)) \phi(\gamma + l - k) \right|^2 \right)^{1/2} \\
&= \left(\sum_{\gamma \in \Gamma_m} \left| \sum_{k \in l + \Omega_m} c_{\epsilon,l;m}(k) \phi(\gamma + l - k) \right| \right. \\
&\quad \left. - \left| \sum_{k \in l + \Omega_m} c(k) \phi(\gamma + l - k) \right| \right)^{1/2} \\
&\leq \left(\sum_{\gamma \in \Gamma_m} \left| \sum_{k \in l + \Omega_m} c_{\epsilon,l;m}(k) \phi(\gamma + l - k) \right| - z_{\epsilon}(\gamma + l) \right)^{1/2} \\
&\quad + \left(\sum_{\gamma \in \Gamma_m} \left| \sum_{k \in l + \Omega_m} c(k) \phi(\gamma + l - k) \right| - z_{\epsilon}(\gamma + l) \right)^{1/2} \\
&\leq 2 \left(\sum_{\gamma \in \Gamma_m} \left| \sum_{k \in l + \Omega_m} c(k) \phi(\gamma + l - k) \right| - z_{\epsilon}(\gamma + l) \right)^{1/2} \\
(6.16) \quad &\leq 2\sqrt{\#\Gamma_m} \|\epsilon\|_{\infty} \leq 2\sqrt{\#\Gamma} \|\epsilon\|_{\infty},
\end{aligned}$$

where the second inequality holds by (4.6) and the last inequality follows from

$$z_{\epsilon}(\gamma + l) = \left| \sum_{k \in l + \Omega_m} c(k) \phi(\gamma + l - k) \right| + \epsilon(\gamma + l), \quad \gamma \in \Gamma_m.$$

From the phase retrievable frame property for $(\phi(\gamma - k))_{k \in K_{A_m}}, \gamma \in \Gamma_m$, we obtain that

$$(6.17) \quad \Omega_m = K_{A_m}, \quad 1 \leq m \leq M.$$

Let $A_{l,m} = \{\gamma \in \Gamma_m : \delta_{\gamma,l;m} = 1\}$. This together with (6.17) and the phase retrievable frame assumption that either $(\phi(\gamma - k))_{k \in \Omega_m}, \gamma \in A_{l,m}$ or $(\phi(\gamma - k))_{k \in \Omega_m}, \gamma \in \Gamma_m \setminus A_{l,m}$ is a spanning set for $\mathbb{R}^{\#\Omega_m}$. This together with (6.16) implies that

$$(6.18) \quad \left(\sum_{k \in l + \Omega_m} |c_{\epsilon,l;m}(k) - \tilde{\delta}_{l,m} c(k)|^2 \right)^{1/2} \leq 2\|\Phi^{-1}\|_2 \sqrt{\#\Gamma} \|\epsilon\|_{\infty}$$

for some sign $\tilde{\delta}_{l,m} \in \{-1, 1\}$.

Now we show that phases of $c_{\epsilon,l;m}, l \in \mathbb{Z}^d, 1 \leq m \leq M$, can be adjusted so that (4.7) holds. Let $\tilde{\delta}_{l,m}, l \in \mathbb{Z}^d, 1 \leq m \leq M$, be as in (6.18). Then for any

$l, l' \in \mathbb{Z}^d$ and $1 \leq m, m' \leq M$, set $\Omega_{l,m;l',m'} = (\Omega_m + l) \cap (\Omega_{l'} + m')$. Then

$$\begin{aligned}
& \langle \tilde{\delta}_{l,m} c_{\epsilon,l;m}, \tilde{\delta}_{l',m'} c_{\epsilon,l';m'} \rangle = \sum_{k \in \Omega_{l,m;l',m'}} \tilde{\delta}_{l,m} \tilde{\delta}_{l',m'} c_{\epsilon,l;m}(k) c_{\epsilon,l';m'}(k) \\
& \geq \sum_{k \in \Omega_{l,m;l',m'}} |c(k)|^2 - \sum_{k \in \Omega_{l,m;l',m'}} |c(k)| |\tilde{\delta}_{l',m'} c_{\epsilon,l';m'}(k) - c(k)| \\
& \quad - \sum_{k \in \Omega_{l,m;l',m'}} |\tilde{\delta}_{l,m} c_{\epsilon,l;m}(k) - c(k)| |c(k)| \\
& \quad - \sum_{k \in \Omega_{l,m;l',m'}} |\tilde{\delta}_{l,m} c_{\epsilon,l;m}(k) - c(k)| |\tilde{\delta}_{l',m'} c_{\epsilon,l';m'}(k) - c(k)| \\
& \geq -\frac{1}{2} \sum_{k \in \Omega_{l,m;l',m'}} \left(|\tilde{\delta}_{l',m'} c_{\epsilon,l';m'}(k) - c(k)|^2 + |\tilde{\delta}_{l,m} c_{\epsilon,l;m}(k) - c(k)|^2 \right) \\
& \quad - \sum_{k \in \Omega_{l,m;l',m'}} |\tilde{\delta}_{l,m} c_{\epsilon,l;m}(k) - c(k)| |\tilde{\delta}_{l',m'} c_{\epsilon,l';m'}(k) - c(k)| \\
(6.19) \quad & \geq -8 \|\Phi^{-1}\|_2^2 \# \Gamma \|\epsilon\|_\infty^2 \geq -M_0,
\end{aligned}$$

where the third inequality follows from (6.18) and the last inequality holds by the assumption (4.13) on the noise level $\|\epsilon\|_\infty$ and the threshold constant M_0 .

The phase adjustments in (4.7) for $c_{\epsilon,l,m}, l \in \mathbb{Z}^d, 1 \leq m \leq M$, are non-unique. Next we show that they are essentially the phase adjustments in (6.19), i.e., for any phase adjustments $\delta_{l,m} \in \{-1, 1\}$ in (4.7) there exists $\delta \in \{-1, 1\}$ such that

$$(6.20) \quad \delta_{l,m} c(k) = \delta \tilde{\delta}_{l,m} c(k) \quad \text{for all } k \in l + \Omega_m.$$

To prove (6.20), we claim that

$$(6.21) \quad \tilde{\delta}_{l,m} / \delta_{l,m} = \tilde{\delta}_{l',m'} / \delta_{l',m'}$$

for all (l, m) and (l', m') with $\Omega_{l,m;l',m'} \cap V_f \neq \emptyset$. Suppose on the contrary that (6.21) does not hold. Then

$$\langle \delta_{l,m} c_{\epsilon,l;m}, \delta_{l',m'} c_{\epsilon,l';m'} \rangle = -\langle \tilde{\delta}_{l,m} c_{\epsilon,l;m}, \tilde{\delta}_{l',m'} c_{\epsilon,l';m'} \rangle.$$

Therefore

$$\begin{aligned}
& \langle \delta_{l,m} c_{\epsilon,l;m}, \delta_{l',m'} c_{\epsilon,l';m'} \rangle \\
& \leq - \sum_{k \in \Omega_{l,m;l',m'}} |c(k)|^2 + \sum_{k \in \Omega_{l,m;l',m'}} |c(k)| |\tilde{\delta}_{l',m'} c_{\epsilon,l';m'}(k) - c(k)| \\
& \quad + \sum_{k \in \Omega_{l,m;l',m'}} |\tilde{\delta}_{l,m} c_{\epsilon,l;m}(k) - c(k)| |c(k)| \\
& \quad + \sum_{k \in \Omega_{l,m;l',m'}} |\tilde{\delta}_{l,m} c_{\epsilon,l;m}(k) - c(k)| |\tilde{\delta}_{l',m'} c_{\epsilon,l';m'}(k) - c(k)| \\
& \leq - \sum_{k \in \Omega_{l,m;l',m'}} |c(k)|^2 + 4\sqrt{\#\Gamma} \|\Phi^{-1}\|_2 \left(\sum_{k \in \Omega_{l,m;l',m'}} |c(k)|^2 \right)^{1/2} \|\epsilon\|_\infty \\
& \quad + 4\#\Gamma \|\Phi^{-1}\|_2^2 \|\epsilon\|_\infty^2 \\
& \leq - \sum_{k \in \Omega_{l,m;l',m'}} |c(k)|^2 + \left(2M_0 \sum_{k \in \Omega_{l,m;l',m'}} |c(k)|^2 \right)^{1/2} + \frac{M_0}{2} < -M_0,
\end{aligned}$$

where the second inequality follows from (6.18), and the third and fourth inequalities hold by (4.10) and (4.13). This contradicts to the requirement (4.7) for the phase adjustment and hence completes the proof of the Claim (6.21).

By (6.21), for any $k \in V_f$ there exist $\delta_k \in \{-1, 1\}$ such that

$$(6.22) \quad \delta_{l,m} c(k) = \delta_k \tilde{\delta}_{l,m} c(k) \quad \text{for all } k \in l + \Omega_m.$$

Let (k_1, k_2) be an edge in \mathcal{G}_f . By Lemma 6.2 there exist $l \in \mathbb{Z}^d$ and $1 \leq m \leq M$ such that $k_1, k_2 \in \Omega_m + l$. Therefore

$$\delta_{l,m} c(k_1) = \delta_{k_1} \tilde{\delta}_{l,m} c(k_1) \quad \text{and} \quad \delta_{l,m} c(k_2) = \delta_{k_2} \tilde{\delta}_{l,m} c(k_2)$$

by (6.22). This implies that $\delta_{k_1} = \delta_{k_2}$ for any edge (k_1, k_2) in \mathcal{G}_f . Combining it with the connected of the graph \mathcal{G}_f , we can find $\delta \in \{-1, 1\}$ such that

$$(6.23) \quad \delta_k = \delta \quad \text{for all } k \in V_f.$$

Combining (6.22) and (6.23) proves (6.20).

By (6.18) and (6.20), we obtain

$$\begin{aligned}
|d_\epsilon(k) - \delta c(k)| & \leq \frac{\sum_{m=1}^M \sum_{l \in \mathbb{Z}^d} |\delta_{l,m} c_{\epsilon,l;m}(k) - \delta c(k)|}{\sum_{m=1}^M \sum_{l \in \mathbb{Z}^d} \chi_{l+\Omega_m}(k)} \\
& = \frac{\sum_{m=1}^M \sum_{l \in \mathbb{Z}^d} |c_{\epsilon,l;m}(k) - \tilde{\delta}_{l,m} c(k)|}{\sum_{m=1}^M \sum_{l \in \mathbb{Z}^d} \chi_{l+\Omega_m}(k)} \\
(6.24) \quad & \leq 2\sqrt{\#\Gamma} \|\Phi^{-1}\|_2 \|\epsilon\|_\infty, \quad k \in \mathbb{Z}^d.
\end{aligned}$$

This together with (4.12) and (4.13) implies that

$$(6.25) \quad |d_\epsilon(k)| \geq \frac{3}{2} \sqrt{M_0} \quad \text{for all } k \in V_f,$$

and

$$(6.26) \quad |d_\epsilon(k)| \leq \frac{1}{2}\sqrt{M_0} \quad \text{for all } k \notin V_f.$$

Combining (4.9), (6.24), (6.25) and (6.26) completes the proof of the desired error estimate (4.14) and (4.15). \square

APPENDIX A. LOCAL COMPLEMENT PROPERTY

A linear space V on \mathbb{R}^d is said to be locally finite-dimensional if it has finite-dimensional restrictions on any bounded open set. Examples of locally finite-dimensional spaces include the space of polynomials of finite degrees, the shift-invariant space generated by finitely many compactly supported functions, and their linear subspaces. The reader may refer [5] and references therein on locally finite-dimensional spaces. In this section, we consider the local complement property for a locally finite-dimensional space, cf. Definition 3.3.

Definition A.1. Let V be a linear space of real-valued continuous signals on \mathbb{R}^d , and $A \subset \mathbb{R}^d$. We say that V has *local complement property on A* if for any $A' \subset A$ there does not exist $f, g \in V$ such that $f, g \not\equiv 0$ on A , $f \equiv 0$ on A' and $g \equiv 0$ on $A \setminus A'$.

In the following theorem, we establish the equivalence between the local complement property on a bounded open set and complement property for ideal sampling functionals on a finite subset.

Theorem A.2. Let A be a bounded open set and V be a locally finite-dimensional space of real-valued continuous signals on \mathbb{R}^d . Then V has the local complement property on A if and only if there exists a finite set $\Gamma \subset A$ such that for any $\Gamma' \subset \Gamma$ either there does not exist $f \in V$ satisfying

$$(A.1) \quad f \not\equiv 0 \text{ on } A \text{ and } f(\gamma') = 0, \gamma' \in \Gamma',$$

or there does not exist $g \in V$ satisfying

$$(A.2) \quad g \not\equiv 0 \text{ on } A \text{ and } g(\gamma) = 0, \gamma \in \Gamma \setminus \Gamma'.$$

The necessity is obvious and the sufficiency follows from the following proposition.

Proposition A.3. Let A and V be as in Theorem A.2. Then there exist a finite set $\Gamma \subset A$ and functions $d_\gamma(x), \gamma \in \Gamma$, such that

$$(A.3) \quad |f(x)|^2 = \sum_{\gamma \in \Gamma} d_\gamma(x) |f(\gamma)|^2, \quad x \in A$$

hold for all $f \in V$.

Proof. Let $g_n, 1 \leq n \leq N$, be a basis of the space $V|_A$, and W be the linear space generated by symmetric matrices $G(x) := (g_n(x)g_{n'}(x))_{1 \leq n, n' \leq N}$, $x \in A$. Then there exists a finite set $\Gamma \subset A$ such that $G(\gamma), \gamma \in \Gamma$, is a basis (or

a spanning set) for the space W . With the above set Γ , we can follow the proof of Theorem 6.1 in Section 6.2 to prove (A.3). \square

Let $g_n, 1 \leq n \leq N$, be a basis of the space $V|_A$, and Γ be as in the proof of Proposition A.3. By Theorem A.2 and [7, Theorem 2.8], we have the following criterion that can be used to verify the local complement property on a bounded open set A in finite steps.

Theorem A.4. *The linear space V has the local complement property on A if and only if for any $\Gamma' \subset \Gamma$, either $(g_n(\gamma'))_{1 \leq n \leq N}, \gamma' \in \Gamma'$ form a frame for \mathbb{R}^N or $(g_n(\gamma))_{1 \leq n \leq N}, \gamma \in \Gamma \setminus \Gamma'$ form a frame for \mathbb{R}^N .*

The local complement property for different open sets can be equivalent. Following the argument used in the proof of Theorem A.2, we have

Proposition A.5. *Let A be a bounded open set and V be a locally finite-dimensional space with the local complement property on A . If B is a bounded open subset of A such that signals g and f satisfying $|g(x)| = |f(x)|$ on B have same magnitude measurements on A , then V has local complement property on B .*

The conclusion in the above proposition is not true in general. For instance, the shift-invariant space $V(\phi_0)$ in Example 2.7 has the local complement property on $(0, 1/2)$, but not on its supset $(0, 1)$.

A linear space may have the local complement property on a bounded open A , but not on some of its open subsets. For instance, one may verify that $V(\phi_1)$ has the local complement property on $(0, 1)$ and on $(-1/2, 1/2)$, but not on their intersection $(0, 1/2)$, where $\phi_1 = \phi_0(2 \cdot)$ and ϕ_0 is given in Example 2.7.

We finish the appendix with a proposition about local linear independence and local complement property.

Proposition A.6. *Let ϕ have local linear independence on any open set. Then there exist $A_m, 1 \leq m \leq M$, such that (3.5) holds and $V(\phi)$ has the local complement property on $A_m, 1 \leq m \leq M$.*

Proof. Let $S_k, k \in \mathbb{Z}^d$, be as in (2.5). For a set $T \subset \mathbb{Z}^d$, define $S_T = \cap_{k \in T} S_k$. We say that T is maximal if $S_T \neq \emptyset$ and $S_{T'} = \emptyset$ for all $T' \subsetneq T$. From the definition, there are finitely many maximal sets T_1, \dots, T_M , and denote the corresponding sets by $A_m := S_{T_m}, 1 \leq m \leq M$.

Clearly (3.5) holds for the above selected open sets as

$$\cup_{m=1}^M T_m = \{k \in \mathbb{Z}^d : S_k \neq \emptyset\}.$$

Then it remains to prove that $V(\phi)$ has local complement property on $A_m, 1 \leq m \leq M$. Assume that $f, g \in V(\phi)$ satisfy $|f(x)| = |g(x)|$ for all $x \in A_m$, which implies that $(f+g)(x)(f-g)(x) = 0$ for all $x \in A_m$. Write $f+g = \sum_{k \in \mathbb{Z}^d} c(k)\phi(\cdot - k)$, $f-g = \sum_{k \in \mathbb{Z}^d} d(k)\phi(\cdot - k)$, and set $B_1 = \{x \in A_m, (f+g)(x) \neq 0\}$ and $B_2 = \{x \in A_m : (f-g)(x) \neq 0\}$. Then either $f-g = 0$ on B_1 , or $f+g = 0$ on B_2 , or $f-g = f+g = 0$ on A_m .

Hence either $c(k) = d(k)$ for all $k \in T_m$ or $c(k) = -d(k)$ on $k \in T_m$ by the local independence on B_1 , or B_2 or A_m . Therefore either $f = g$ on A_m , or $f = -g$ on A_m , or $f = g = 0$ on A_m . This completes the proof. \square

REFERENCES

- [1] R. Alaifari, I. Daubechies, P. Grohs and G. Thakur, Reconstructing real-valued functions from unsigned coefficients with respect to wavelet and other frames, *J. Fourier Anal. Appl.*, 2016, to appear.
- [2] R. Alaifari, I. Daubechies, P. Grohs and R. Yin, Stable phase retrieval in infinite dimensions, Arxiv preprint, arXiv:1609.00034
- [3] R. Alaifari and P. Grohs, Phase retrieval in the general setting of continuous frames for Banach spaces, Arxiv preprint, arXiv:1604.03163
- [4] A. Aldroubi and K. Gröchenig, Non-uniform sampling in shift-invariant space, *SIAM Rev.*, **43**(2001), 585–620.
- [5] A. Aldroubi and Q. Sun, Locally finite dimensional shift-invariant spaces in \mathbb{R}^d , *Proc. Amer. Math. Soc.*, **130**(2002), 2641–2654.
- [6] A. Aldroubi, Q. Sun and W.-S. Tang, Convolution, average sampling, and Calderon resolution of the identity of shift-invariant spaces, *J. Fourier Anal. Appl.*, **11**(2005), 215–244.
- [7] R. Balan, P. G. Casazza and D. Edidin, On signal reconstruction without phase, *Appl. Comp. Harm. Anal.*, **20**(2006), 345–356.
- [8] R. Balan, B. G. Bodmann, P. G. Casazza and D. Edidin, Painless reconstruction from magnitudes of frame coefficients, *J. Fourier Anal. Appl.*, **15**(2009), 488–501.
- [9] A. S. Bandeira, J. Cahill, D. G. Mixon, and A. A. Nelson, Saving phase: injectivity and stability for phase retrieval, *Appl. Comput. Harmon. Anal.*, **37**(2014), 106–125.
- [10] A. Ben-Artzi and A. Ron, On the integer translates of a compactly supported function: dual bases and linear projectors, *SIAM J. Math. Anal.*, **21**(1990), 1550–1562.
- [11] M. Bownik, The structure of shift-invariant subspaces of $L^2(\mathbb{R}^d)$, *J. Funct. Anal.*, **177**(2000), 282–309.
- [12] J. Cahill, P. G. Casazza and I. Daubechies, Phase retrieval in infinite-dimensional Hilbert spaces, *Trans. Amer. Math. Soc.*, Ser. B, **3**(2016), 63–76.
- [13] E. Candes, T. Strohmer, and V. Voroninski, Phaselift: exact and stable signal recovery from magnitude measurements via convex programming, *Comm. Pure Appl. Math.*, **66**(2013), 1241–1274.
- [14] E. J. Candes, Y. C. Eldar, T. Strohmer and V. Voroninski, Phase retrieval via matrix completion, *SIAM J. Imaging Sci.*, **6**(2013), 199–225.
- [15] P. G. Casazza, D. Ghereishi, S. Jose and J. C. Tremain, Norm retrieval and phase retrieval by projections, Arxiv preprint, arXiv:1701.08014
- [16] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM 1992.
- [17] C. de Boor, R. A. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$, *J. Funct. Anal.*, **119**(1994), 37–78.
- [18] C. de Boor and K. Höllig, B-splines from parallelepipeds, *J. Anal. Math.*, **62**(1983), 99–115.
- [19] C. de Boor, K. Höllig and S.D. Riemenschneider, *Box Splines*, Springer-Verlag, Berlin, 1993.
- [20] Y. Chen, C. Cheng, Q. Sun and H. Wang, Phase retrieval of real-valued signals in a shift-invariant space, Arxiv preprint, arXiv:1603.01592
- [21] C. Cheng, Y. Jiang and Q. Sun, Spatially distributed sampling and reconstruction, Arxiv preprint, arXiv:1511.08541
- [22] W. Dahmen and C. A. Micchelli, On the local linear independence of translates of a box spline, *Studia Math.*, **82**(1985), 243–263.

- [23] W. Dahmen and C. A. Micchelli, Translates of multivariate splines, *Linear Algebra Appl.*, **52**(1982), 217–234.
- [24] J. R. Fienup, Reconstruction of an object from the modulus of its Fourier transform, *Opt. Lett.*, **3**(1978), 27–29.
- [25] B. Gao, Q. Sun, Y. Wang and Z. Xu, Phase retrieval from the magnitudes of affine linear measurements, Arxiv preprint, arXiv:1608.06117
- [26] P. Hand and V. Voroninski, Corruption robust phase retrieval via linear programming, Arxiv preprint, arXiv:1612.03547
- [27] M. A. Iwen, B. Preskitt, R. Saab and A. Viswanathan, Phase retrieval from local measurements: improved robustness via eigenvector-based angular synchronization, Arxiv preprint, arXiv:1612.01182
- [28] K. Jaganathan, Y. C. Eldar and B. Hassibi, Phase retrieval: an overview of recent developments, In *Optical Compressive Imaging*, edited by A. Stern, CRC Press, 2016, pp. 261–296.
- [29] R.-Q. Jia, Local linear independence of the translates of a box spline, *Constr. Approx.*, **1**(1985), 175–182.
- [30] R.-Q. Jia and C. A. Micchelli, On linear independence of integer translates of a finite number of functions, *Proc. Edinburgh Math. Soc.*, **36**(1992), 69–75.
- [31] P. G. Lemarié, Fonctions á support compact dans les analyses multirésolutions, *Rev. Mat. Iberoamericana*, **7**(1991), 157–182.
- [32] L. Li, C. Cheng, D. Han, Q. Sun and G. Shi, Phase retrieval from multiple-window short-time Fourier measurements, *IEEE Signal Process. Lett.*, accepted
- [33] S. Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*, Academic Press, 2009.
- [34] S. Mallat and I. Waldspurger, Phase retrieval for the Cauchy wavelet transform, *J. Fourier Anal. Appl.*, **21**(2014), 1–59.
- [35] Y. Meyer, Ondelettes sur l’intervalle, *Rev. Mat. Iberoamericana*, **7**(1991), 115–133.
- [36] V. Pohl, F. Yang and H. Boche, Phase retrieval from low-rate samples, *Sampl. Theory Signal Image Process.*, **13**(2014), 71–99.
- [37] V. Pohl, F. Yang and H. Boche, Phaseless signal recovery in infinite dimensional spaces using structured modulations, *J. Fourier Anal. Appl.*, **20**(2014), 1212–1233.
- [38] A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, *Const. Approx.*, **5**(1989), 297–308.
- [39] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao and M. Segev, Phase retrieval with application to optical imaging: a contemporary overview, *IEEE Signal Proc. Mag.*, **32**(2015), 87–109.
- [40] B. A. Shenoy, S. Mulleti and C. S. Seelamantula, Exact phase retrieval in principal shift-invariant spaces, *IEEE Trans. Signal Proc.*, **64**(2016), 406–416.
- [41] Q. Sun, Local reconstruction for sampling in shift-invariant spaces, *Adv. Comput. Math.*, **32**(2010), 335–352.
- [42] Q. Sun, Nonuniform average sampling and reconstruction of signals with finite rate of innovation, *SIAM J. Math. Anal.*, **38**(2006), 1389–1422.
- [43] Q. Sun, A note on the integer translates of a compactly supported distribution on \mathbb{R} , *Archiv Math.*, **60**(1993), 359–363.
- [44] G. Thakur, Reconstruction of bandlimited functions from unsigned samples, *J. Fourier Anal. Appl.*, **17**(2011), 720–732.
- [45] M. Unser, Splines: a perfect fit for signal and image processing, *IEEE Signal Proc. Mag.*, **16**(1999), 22–38.
- [46] G. Wahba, *Spline Models for Observational Data*, SIAM, 1990.

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